

ZERO (SUB-)SEQUENCES OF ENTIRE FUNCTIONS

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1. ZERO SUBSETS FOR WIEGHTED CLASSES. Let D be a domain of the complex plane \mathbb{C} . Denote by $\text{sbh}(D)$ the class of subharmonic functions in D , and $\text{sbh}(D) \ni -\infty: z \mapsto -\infty$, $z \in D$ (see [1]).

A function $V \in \text{sbh}(\mathbb{C} \setminus \{0\})$ is called a Jensen potential if there exists a $R_V > 0$ such that $0 \leq V(z) \leq \max\{0, \log(R_V/|z|)\}$ for all $z \in \mathbb{C} \setminus \{0\}$. The class of all Jensen potentials will be denoted by PJ_0 . Let $M \in \text{sbh}(\mathbb{C})$ with $M \neq -\infty$, and let $\nu_M := \frac{1}{2\pi} \Delta M$ be the Riesz measure for M , where Δ is the Laplace operator on distributions. Let $Z = \{z_k\}_{k=1,2}, \subset \mathbb{C} \setminus \{0\}$ be a sequence of points.

Theorem 1. *If there are a non-zero entire function f (we write $f \neq 0$) vanishes on Z (we write $f(Z) = 0$) and $r_0 > 0$ such that $|f| \leq e^M$ on \mathbb{C} (pointwise), and $\inf_{|z| < r_0} M(z) > -\infty$, then*

$$\sup \left\{ \sum_k V(z_k) - \int_{\mathbb{C} \setminus \{0\}} V d\nu_M : V \in PJ_0 \right\} < +\infty.$$

Conversely, if

$$\sup \left\{ \sum_k V(z_k) - \int_{\mathbb{C} \setminus \{0\}} V d\nu_M : V \in PJ_0 \cap C^\infty(\mathbb{C} \setminus \{0\}) \right\} < +\infty,$$

then, for each number $N > 0$, there is an entire function $f \neq 0$ such that $f(Z) = 0$ and

$$|f(z)| \leq \exp \left(\frac{1}{2\pi} \int_0^{2\pi} M \left(z + \frac{1}{1+|z|^N} e^{i\theta} \right) d\theta \right) \quad \text{for all } z \in \mathbb{C}.$$

This Theorem 1 improves our earlier results from [2]–[8] etc.

2. SUBSEQUENCES OF ZEROS FOR FUNCTIONS WITH MAJORANT OF CARTWRIGHT CLASS. A function $M \in \text{sbh}(\mathbb{C})$ belong to the (Cartwright) class \mathcal{C} (see [9]) if M is harmonic outside the real axis \mathbb{R} , $M(0) = 0$, $M(z) = M(\bar{z})$ for all $z \in \mathbb{C}$, and

$$\limsup_{z \rightarrow \infty} \frac{M(z)}{|z|} < +\infty, \quad \int_{\mathbb{R}} \frac{\max\{0, M(x)\}}{x^2} dx < +\infty.$$

Let ν_M be the Riesz measure of $M \in \mathcal{C}$. We denote by $\nu_M^{\mathbb{R}}$ its distribution function

$$\nu_M^{\mathbb{R}}(t) := \begin{cases} -\nu_M([t, 0)) & \text{when } t < 0, \\ \nu_M([0, t]) & \text{when } t \geq 0, \end{cases} \quad t \in \mathbb{R}.$$

We introduce the class $R\mathcal{P}_0$ of test functions as the subclass of all upper semicontinuous functions $\phi: \mathbb{R} \setminus \{0\} \rightarrow [0, +\infty)$ such that

$$\phi(x) \equiv 0 \text{ for } |x| \geq R_\phi, \quad \limsup_{0 \neq x \rightarrow 0} \frac{\phi(x)}{-\log|x|} \leq 1,$$

and, for each $x_0 \in \mathbb{R} \setminus \{0\}$, there is $r_0 \in (0, |x_0|)$ such that

$$\phi(x_0) \leq \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \phi(x_0 + x) \frac{1}{x} \log \left| \frac{x+r}{x-r} \right| dx \quad \text{for all } r \in (0, r_0).$$

Theorem 2. *Let $M \in \mathcal{C}$, $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{C} \setminus \{0\}$.*

If there exists an entire function $f \neq 0$ such that $f(Z) = 0$ and $|f| \leq e^M$ on \mathbb{C} (pointwise), then

$$\sup \left\{ \sum_k (P\phi)(z_k) - \int_{-\infty}^{+\infty} \phi(t) d\nu_M^{\mathbb{R}}(t) : \phi \in R\mathcal{P}_0 \right\} < +\infty,$$

where $(P\phi)(z) := \phi(z)$ for $\Im z = 0$, and, for $\Im z \neq 0$,

$$(P\phi)(z) := \frac{1}{\pi} \int_{-\infty}^{+\infty} \left| \Im \frac{1}{x-z} \right| \phi(x) dx \quad (\text{the Poisson integral}).$$

Conversely, if

$$\sup \left\{ \sum_k (P\phi)(z_k) - \int_{-\infty}^{+\infty} \phi(t) d\nu_M^{\mathbb{R}}(t) : \phi \in R\mathcal{P}_0 \cap C^\infty(\mathbb{R} \setminus \{0\}) \right\} < +\infty,$$

then, for each $N > 0$, there exists an entire function $f \neq 0$ such that $f(Z) = 0$ and $|f| \leq \exp M_N$ on \mathbb{C} where

$$M_N(z) := \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} M\left(z + \frac{1}{1+|\Re z|^N} e^{i\theta}\right) d\theta & \text{when } |\Im z| \leq |\Re z|^{-N}, \\ M(z) & \text{when } |\Im z| \geq |\Re z|^{-N}. \end{cases}$$

This Theorem 2 generalizes our previous results [10]–[11].

3. (NON-)UNIQUENESS SEQUENCES. For a subset $S \subset \mathbb{C}$, we denote by $\text{sbh}(S)$ the class of functions that are subharmonic on some open set containing S . We set

$$\text{sbh}^+(S) := \{u \in \text{sbh}(S) : u(z) \geq 0 \text{ for all } z \in S\}.$$

Given $r > 0$ and $b > 0$, we set

$$D(r) := \{z \in \mathbb{C} : |z| < r\}, \quad \overline{D(r)} := \{z \in \mathbb{C} : |z| \leq r\},$$

$$\text{sbh}_0^+(r; \leq b) := \left\{ v \in \text{sbh}^+(\mathbb{C} \setminus D(r)) : \lim_{z \rightarrow \infty} v(z) = 0, \sup_{|z|=r} v(z) \leq b \right\}.$$

We denote by $\text{const}_{a_1, a_2, \dots}$ a constant depending only on a_1, a_2, \dots .

Theorem 3. *Let $M \in \text{sbh}(\mathbb{C})$ with the Riesz measure ν_M , $r_0, b > 0$. Then there are numbers $C := \text{const}_{r_0, b} > 0$, $\overline{C}_M := \text{const}_{r_0, M} \geq 0$ such that, for any $v \in \text{sbh}_0^+(r_0; \leq b)$ and for each function $u \in \text{sbh}(\mathbb{C}) \setminus \{-\infty\}$ with the Riesz measure ν_u , the pointwise inequality $u \leq M$ on $\mathbb{C} \setminus D(r_0)$ entails the inequality*

$$\int_{\mathbb{C} \setminus D(r_0)} v d\nu_u \leq \int_{\mathbb{C} \setminus D(r_0)} v d\nu_M + C \overline{C}_M - Cu(z_0), \quad (\text{C})$$

where a constant $\overline{C}_M < +\infty$ is positively homogeneous of M , i. e. $\overline{C}_{aM} = a\overline{C}_M$ for $a \in [0, +\infty)$, and upper semi-additive of M , i. e. $\overline{C}_{M_1+M_2} \leq \overline{C}_{M_1} + \overline{C}_{M_2}$ for $M_1, M_2 \in \text{sbh}(\mathbb{C})$. Besides, the integral in the right parts of the inequality (C) can be replaced by the integral $\int_{\mathbb{C} \setminus D(r_0)} M d\nu_v$,

where ν_v is the Riesz measure of v .

COROLLARY 1. *Let $M \in \text{sbh}(\mathbb{C})$ with the Riesz measure ν_M , and f be an entire function. Suppose that there is a number $r_0 > 0$ such that $\log |f| \leq M$ on $\mathbb{C} \setminus D(r_0)$, and f vanishes on a sequence $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{C} \setminus D(r_0)$. If $v \in \text{sbh}^+(\mathbb{C} \setminus D(r_0))$, $\lim_{z \rightarrow \infty} v(z) = 0$, and*

$$\int_{\mathbb{C} \setminus D(r_0)} v d\nu_M < +\infty, \quad \text{but} \quad \sum_k v(z_k) = +\infty,$$

then $f = 0$.

COROLLARY 2. *Let $r_0 > 0$, and $v \in \text{sbh}^+(\mathbb{C} \setminus D(r_0))$ with the Riesz measure ν_v satisfies the condition $\lim_{z \rightarrow \infty} v(z) = 0$. If an entire function $f \neq 0$ vanishes on a sequence $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{C}$, and f satisfies the condition $\int_{\mathbb{C} \setminus D(r_0)} \log |f| d\nu_v < +\infty$, then*

$$\sum_{z_k \in D \setminus D_0} v(z_k) < +\infty.$$

4. RADIAL VERSIONS. Let $r_0 \in (0, +\infty)$. Let $M: (r_0, +\infty) \rightarrow \mathbb{R}$ be an increasing continuous function. Further assume that there exists its left-hand derivative M'_{left} on $(r_0, +\infty)$ and the function $t \mapsto tM'_{\text{left}}(t)$ is increasing on $(r_0, +\infty)$.

Let $q: [r_0, +\infty) \rightarrow \mathbb{R}$ be a bounded positive decreasing function. Further assume that $\int_{r_0}^{+\infty} q(t) \frac{dt}{t} < +\infty$.

Let $f \neq 0$ be an entire function such that f vanishes on a sequence $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{C} \setminus \overline{D(r_0)}$.

COROLLARY 3. *If $|f(z)| \leq \exp M(|z|)$ for all $|z| > r_0$ and*

$$\int_{r_0}^{+\infty} q(t) M'_{\text{left}}(t) dt < +\infty,$$

then

$$\sum_{|z_k| > r_0} \int_{|z_k|}^{+\infty} \frac{q(t)}{t} dt < +\infty. \quad (q)$$

COROLLARY 4. If $\int_{r_0}^{+\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} \log |f(te^{i\theta})| d\theta \right) dq(t) < +\infty$, then the relation (q) is fulfilled.

These Corollaries 3,4 improve a part of results from [2]–[8], [12]. Explicit non-radial cases are much more complicated and will be discussed in detail in another place.

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References

1. Ransford Th. *Potential Theory in the Complex Plane* – Cambridge: Cambridge University Press, 1995.
2. Хабибуллин Б. Н. *Полнота систем экспонент и множества единственности* (издание 4^{ое}, дополненное). – Уфа: РИЦ БашГУ. 2012.
http://www.researchgate.net/profile/Bulat_Khabibullin/contributions
3. Khabibullin B. N. *Sets of uniqueness in spaces of entire functions of a single variable* // Mathematics of the USSR–Izvestiya. – 1992. V. 39. – no. 2. – P. 1063–1084. — Изв. АН СССР. Сер. матем. – 1991. – Т. 55. – № 5. – С. 1101–1123.
4. Khabibullin B. N. *Distribution of zeros of entire functions and the balayage*. Diss. of Dr. of ph.-math. Sci. – 1993. Kharkiv.
5. Khabibullin B. N. *Dual representation of superlinear functionals and its applications in function theory*. II // Izvestiya: Mathematics. – 2001. – V. 65. – no. 5. – P. 1017–1039. — Изв. РАН. Сер. матем. – 2001. – Т. 65. – № 5. – С. 167–190.
6. Khabibullin B. N. *Zero sequences of holomorphic functions, representation of meromorphic functions, and harmonic minorants* // Sbornik: Mathematics. – 2007. – 198:2. – P. 261–298. — Матем. сб. – 2007. – Т. 198. – № 2. – С. 121–160.
7. Khabibullin B. N., Khabibullin F. B., Cherednikova L. Yu. *Zero subsequences for classes of holomorphic functions: stability and the entropy of arcwise connectedness*. II // St. Petersburg Math. Journal. – 2009. – 20:1. – P. 131–162. — Алгебра и анализ. 2008. Т. 20. № 1. С. 190–236.
8. Khabibullin B. N. *Sequences of non-uniqueness for weight spaces of holomorphic functions* // Russian Mathematics (Izvestiya VUZ. Matematika) – 2015 – 59:4. – P. 63–70. — Известия вузов. Математика. – 2015. – № 4. – С. 75–84.
9. Matsaev V., Sodin M. *Distribution of Hilbert transforms of measures* // Geom. funct. anal. – 2000. – V. 10. – P. 160–184.
10. Khabibullin B. N., Talipova G. R., Khabibullin F. B. *Zero subsequences for Bernstein's spaces and the completeness of exponential systems in spaces of functions on an interval* // St. Petersburg Mathematical Journal. – 2015 – 26:2. – P. 319–340. — Алгебра и анализ. – 2014. – Т. 26, № 2. – С. 185–215.
11. Talipova G. R., Khabibullin B. N. *Sequences of uniqueness for classes of entire functions of exponential type with restrictions on the real axis* // Bulletin of Bashkir University. – 2015. – V. 20. – no. 1 – P. 1–5 — Вестник Башкирского ун-та. – 2015. – Т. 20. – № 1. – С. 1–5.
12. Bykov S. V., Shamoyan F. A. *On zero of entire functions with majorant of infinite order* // St. Petersburg Mathematical Journal. – 2010 – 21:6. – P. 893–901. — Алгебра и анализ – 2009. — Т. 21. – № 6. – С. 66–79.